SERIES WORKSHEET 2

Problem 1. Find the radius of convergence and the interval of convergence (for (6) and (8) just the radius suffices).

(1) $\displaystyle\sum_{n=1}^{\infty} \frac{x^{3n}}{2^n - 3^n}$, (2) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n^4 4^n}$, (3) $\displaystyle\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^2+n-1}$, (4) $\displaystyle\sum_{n=1}^{\infty} \sqrt{n+4^n}\, x^n$,

(5) $\displaystyle\sum_{n=1}^{\infty} \frac{(x-3)^{2n}}{n^3}$, (6) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n} x^n$, (7) $\displaystyle\sum_{n=1}^{\infty} \frac{x^{n^2}}{n}$, (8) $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^k}{(kn)!} x^n \ (k \in \mathbb{Z}_{>0})$.

Problem 2. Compute the values of the sums:

(1) $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{n!}$, (2) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{3^n}$, (3) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{ne^n}$, (4) $\displaystyle\sum_{n=0}^{\infty} \pi^{2n} \frac{(-1)^n}{(2n+1)!}$,

(5) $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{8^n (n!)^2}$ (Hint: Show that $\displaystyle\binom{-\frac{1}{2}}{n} = (-4)^{-n} \frac{(2n)!}{(n!)^2}$), (6) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n)!}$.

(7) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)3^n}$.

Problem 3. Express the given functions as power series centered at 0.

(1) $\displaystyle\frac{x^2}{x^4+16}$, (2) $\displaystyle\frac{1+x}{1-x}$, (3) $\sin^2(x)$, (4) $(x+1)e^{x^2}$.

Problem 4. Suppose $f(x) = \displaystyle\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence 1 and $\displaystyle\sum_{n=0}^{\infty} a_n$ converges. *Abel's theorem* says that then $\displaystyle\lim_{x \to 1^-} f(x) = \sum_{n=0}^{\infty} a_n$. Use this to compute the following sums:

(1) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$, (2) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1}$.

Problem 5. The *Bernoulli numbers* B_n are defined by the power series expansion

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n.$$

Compute B_n for $n = 0, 1, 2, 3, 4, 5, 6$. Show that $\displaystyle\frac{x}{e^x-1} + \frac{x}{2}$ is even and hence deduce that $B_n = 0$ whenever $n > 1$ is odd.

Problem 6. Recall that the Fibonacci numbers are defined recursively by $F_0 = 0, F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for $n \geq 0$. We can use power series to derive the explicit formula for F_n as follows. Let $f(x) = \sum_{n=0}^{\infty} F_n x^n$.

(1) Use the recurrence relation and the initial conditions for F_n to deduce $f(x) = \frac{-x}{x^2 + x - 1}$.

(2) Use partial fraction decomposition to write $f(x)$ as $f(x) = \frac{A}{x-\alpha} + \frac{B}{x-\beta}$ for suitable numbers A, B, α, β.

(3) Use the expression found for f in (2) and the geometric series to deduce

$$F_n = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n \right)$$

by comparing coefficients.

Problem 7. Use the first order Taylor polynomial and its error bound to show the error bound for the midpoint rule (Hint: First consider one interval $[x_0, x_1]$. Using the Taylor inequality for $|f(x) - T_1(x)|$ show that $\left| \int_{x_0}^{x_1} f(x) \mathrm{d}x - \Delta x f(\overline{x_1}) \right| \leq \frac{(\Delta x)^3 M}{24}$ where M is a bound for $|f''|$. Then add up all the error terms for the individual intervals $[x_i, x_{i+1}]$ to get the error bound on $[a, b]$).

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